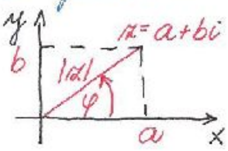
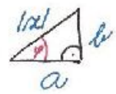
7. Goniometrický tvar komplexního čísla
(exponenciální tvar)

Př.) Gaussova rovina





$\sin\varphi = \frac{b}{|z|} \Rightarrow b = |z|\sin\varphi$ $\qquad |z| = \sqrt{a^2+b^2}$ ... abs. hodnota

$\cos\varphi = \frac{a}{|z|} \Rightarrow a = |z|\cos\varphi$

$$z = a+bi = |z|\cos\varphi + i|z|\sin\varphi = |z|\underbrace{(\cos\varphi + i\sin\varphi)}_{\text{goniometrický tvar}}$$

DEF.

GONIOMETRICKÝM TVAREM nenulového komplexního čísla z ($z \neq 0$) nazýváme jeho vyjádření ve tvaru $z = |z|(\cos\varphi + i\sin\varphi)$,

kde $|z|$ je jeho absolutní hodnota a číslo φ nazýváme argument komplexního čísla (velikost orientovaného úhlu, který svírá průvodič bodu $Z[a;b]$ s kladnou poloosou x)

- GONIOMETRICKÝ TVAR není určen jednoznačně

  [$\cos\varphi = \cos(\varphi + 2k\pi)$ $\quad \sin\varphi = \sin(\varphi + 2k\pi)$ $\quad k \in Z$ – periodické fce]

[EXPONENCIÁLNÍ TVAR – zkrácený zápis goniometrického tvaru

$z = |z|e^{i\varphi} \qquad e^{i\varphi} = \cos\varphi + i\sin\varphi$]

- ROVNOST NENUL. KOMPL. ČÍSEL v goniom. tvaru

  $z_1 = |z_1|(\cos\varphi_1 + i\sin\varphi_1) \qquad z_2 = |z_2|(\cos\varphi_2 + i\sin\varphi_2)$

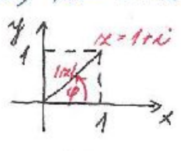
  $z_1 = z_2 \Leftrightarrow |z_1| = |z_2| \wedge \varphi_1 = \varphi_2 + 2k\pi \quad k \in Z$

- KOMPLEXNÍ JEDNOTKY: $|z| = 1 \Rightarrow z = \cos\varphi + i\sin\varphi$

Příklady

① Převeďte na goniometrický tvar (musíme určit $|z|$ a φ)

a) $z = 1+i = [1,1]$

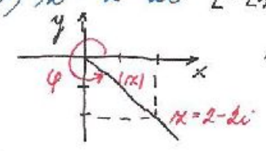


$|z| = \sqrt{1^2+1^2} = \sqrt{2}$

$\varphi = 45° = \frac{\pi}{4}$

$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \left[= \sqrt{2}\,e^{i\frac{\pi}{4}}\right]$

b) $z = 2-2i$ $[= [2;-2]]$



$|z| = \sqrt{2^2+(-2)^2} = \sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}$

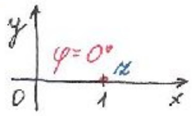
$\varphi = 360° - 45° = 315°$

$\varphi = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

$z = 2\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) \left[= 2\sqrt{2}\,e^{i\frac{7\pi}{4}}\right]$

někdy i $z = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) \left[= 2\sqrt{2}\,e^{-i\frac{\pi}{4}}\right]$

c) $z = i^{80} = i^{4\cdot 20+0} = i^0 = 1$

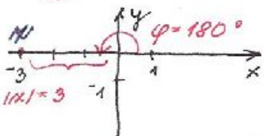


$|z| = 1$ kompl. jedn.

$\varphi = 0° = 0$ rad

$\underset{\sim\sim\sim}{z = \cos 0 + i \sin 0}$ $(= e^{i0})$

d) $z = -3$ DÚ

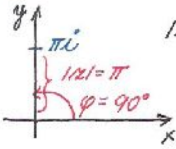


$|z| = |-3| = 3$

$\varphi = 180° = \pi$

$\underset{\sim\sim\sim}{z = 3(\cos\pi + i\sin\pi)}$ $(= 3e^{i\pi})$

e) $z = \pi i$



$|z| = |\pi i| = \sqrt{\pi^2} = \pi$

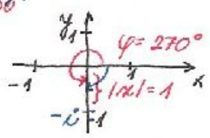
$0 + \pi i$

$\varphi = 90° = \frac{\pi}{2}$

$\underset{\sim\sim\sim}{z = \pi\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}$

$(= \pi e^{i\frac{\pi}{2}})$

f) $z = -i$ DÚ



$|z| = |-i| = \sqrt{0^2 + (-1)^2} = 1$ kompl. jedn.

$0 - i$

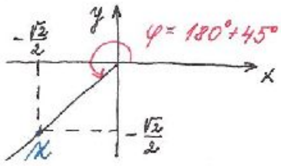
$\varphi = 270° = \frac{3\pi}{2}$

$(\varphi = -90° = -\frac{\pi}{2})$

$\underset{\sim\sim\sim}{z = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}}$ $(= e^{i\frac{3\pi}{2}})$

$[z = \cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2})]$ $(= e^{-i\frac{\pi}{2}})$

g) $z = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$



$|z| = \sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2}$

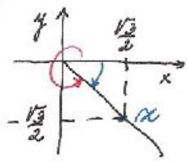
$|z| = \sqrt{\frac{2}{4} + \frac{2}{4}} = \sqrt{\frac{4}{4}} = 1$ kompl. jedn.

$\varphi = 180° + 45°$

$\varphi = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$

$\underset{\sim\sim\sim}{z = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}}$

h) $z = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}i$ DÚ



$|z| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2}$

$|z| = \sqrt{\frac{3}{4} + \frac{3}{4}} = \sqrt{\frac{6}{4}}$

$|z| = \frac{\sqrt{6}}{2}$

$\varphi = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

$(\varphi = 360° - 45° = 315°)$

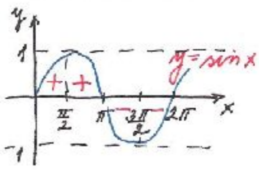
$\underset{\sim\sim\sim}{z = \frac{\sqrt{6}}{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)}$

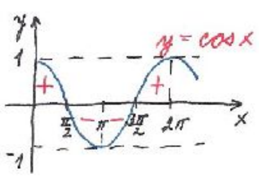
$\left(z = \frac{\sqrt{6}}{2}\left(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right)\right)$

ch) $(1-i)^2 + 2i = 1 - 2i + i^2 + 2i = 0$

$z = 0$ GONIOM. TVAR pro $z = 0$ NEEXISTUJE

GONIOMETRICKÉ FUNKCE





	$0°$	$30°$	$45°$	$60°$	$90°$
	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

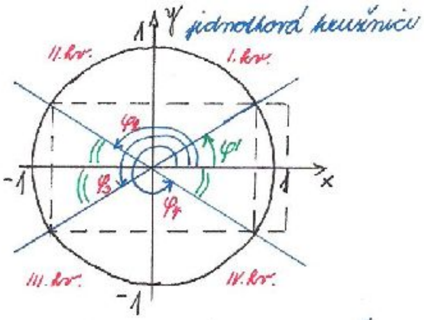
$360° \ldots 2\pi$ rad

$180° \ldots \pi$ rad

$90° \ldots \frac{\pi}{2}$ rad

$60° \ldots \frac{\pi}{3}$ rad

$30° \ldots \frac{\pi}{6}$ rad



I. kv. $\varphi_1 = \varphi'$

II. kv. $\varphi_2 = 180° - \varphi'$

$\varphi_2 = \pi - \varphi'$

III. kv. $\varphi_3 = 180° + \varphi'$

$\varphi_3 = \pi + \varphi'$

IV. kv. $\varphi_4 = 360° - \varphi'$

$\varphi_4 = 2\pi - \varphi'$

φ' … pomocný úhel z I. kv.

(gon. fce vždy v I. kv. ⊕ > 0)

PARITA (SUDOST, LICHOST)

$\sin(-x) = -\sin x$ (LICHÁ)

! $\cos(-x) = \cos x$ (SUDÁ)

PERIODIČNOST

$\sin x = \sin(x + 2k\pi)$

$\cos x = \cos(x + 2k\pi)$

$k \in \mathbb{Z}$

OPAKOVÁNÍ

př. $\cos(-120°) = \cos 120° = -\cos 60° = -\frac{1}{2}$

[$\cos(-x) = \cos x$] využ. PARITU (sudost, lichost)

II. kv. $\ominus$

$120° = 180° - \varphi'$

$\varphi' = 60°$

$\sin(-300°) = -\sin 300° = -(-\sin 60°) = \sin 60° = \frac{\sqrt{3}}{2}$

[$\sin(-x) = -\sin x$] využ. paritu (sudost, lichost)

IV. kv. $\ominus$

$300° = 360° - \varphi'$

$\varphi' = 60°$

nebo $= \sin(-300° + 1.360°) = \sin 60° = \frac{\sqrt{3}}{2}$

využ. periodičnost $\sin x = \sin(x + k \cdot 360°)$

$\sin\left(-\frac{5\pi}{6}\right) = \sin\left(-\frac{5\pi}{6} + 2\pi\right) = \sin\left(-\frac{5\pi}{6} + \frac{12\pi}{6}\right) = \sin\frac{7\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) =$ 3. kv. $\ominus$ $\pi + \varphi'$

$= -\sin\frac{\pi}{6} = -\sin 30° = -\frac{1}{2}$ (OBTÍŽNĚJŠÍ) φ'

– JEDNOD. => na stupně

$\sin\left(-\frac{5\pi}{6}\right) = -\sin\frac{5\pi}{6} = -\sin 150° = -\sin 30° = -\frac{1}{2}$

$\left(\frac{5\pi}{6} = 5 \cdot \frac{\pi}{6} = 5 \cdot 30° = 150°\right)$

II. kv. $\oplus$

$150° = 180° - \varphi'$

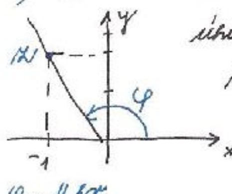
$\varphi' = 30°$

[lze různě]

② Zapište v goniometrickém tvaru

48/44

a) $z = -1 + i\sqrt{3}$ $[-1, \sqrt{3}]$ 47



$\varphi \ldots$ II. kv.

úhel nelze určit zpaměti z obr. => φ musíme VYPOČÍTAT (ale z obr. si lze určit KVADRANT)

$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$

$\cos\varphi = \frac{a}{|z|} = \frac{-1}{2}$ $\ominus$ => II., III. kv.

$\sin\varphi = \frac{b}{|z|} = \frac{\sqrt{3}}{2}$ $\oplus$ => I., II. kv.

} II. kv.

φ vidíš z obr., že $\varphi \ldots$ II. kv.

pomocný úhel φ': $\sin\varphi' = +\frac{\sqrt{3}}{2}$ z 1. kv. $\varphi' = 60° = \frac{\pi}{3}$

=> II. kv. $\varphi = \pi - \varphi' = \pi - \frac{\pi}{3}$

$\varphi = \frac{2\pi}{3}$

$z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

b) $z = 1 + \underbrace{\cos\frac{1}{3}\pi}_{\frac{1}{2}} + i\underbrace{\sin\frac{1}{3}\pi}_{\frac{\sqrt{3}}{2}}$ (60°, 60°) ! NENÍ GONIOM. TVAR !

58/45

na alg. tvar

$z = 1 + \frac{1}{2} + \frac{\sqrt{3}}{2}i = \frac{3}{2} + \frac{\sqrt{3}}{2}i$ $\left[\frac{3}{2}, \frac{\sqrt{3}}{2}\right]$ => z leží v 1. kv. => úhel φ z 1. kv.

$|z| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{\frac{12}{4}} = \sqrt{3}$

[$\cos\varphi = \frac{a}{|z|} = \frac{\frac{3}{2}}{\sqrt{3}} = \frac{3}{2\sqrt{3}} = \frac{3}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{2 \cdot 3} = \frac{\sqrt{3}}{2}$

$\sin\varphi = \frac{b}{|z|} = \frac{\frac{\sqrt{3}}{2}}{\sqrt{3}} = \frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2}$ => $\sin\varphi = \frac{1}{2} \wedge \varphi \ldots$ 1. kv.

1. kv. $\varphi = \varphi'$ $\varphi' = 30° = \frac{\pi}{6}$ $(\sin\varphi' = \frac{1}{2})$

$\varphi = \frac{\pi}{6}$

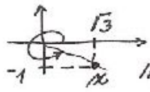
$z = \sqrt{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

[pokud neumíš určit, v kterém kvadrantu leží z => musíš jako a) modrou čárkovanou část]

[hodnotu φ' lze určit i z $\cos\varphi' = \frac{\sqrt{3}}{2}$, $\varphi' = 30°$

– vždy pro φ' bereme kladnou hodnotu gon. fce]

c) $z = \sqrt{3} - i \quad [\sqrt{3}, -1]$



IV. kv. φ

$|z| = \sqrt{3+1} = 2$

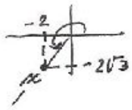
$\sin\varphi = \frac{b}{|z|} = -\frac{1}{2} \quad \left[\begin{array}{l} \sin\varphi' = \frac{1}{2} \\ \varphi' = 30° = \frac{\pi}{6} \end{array}\right] \Rightarrow \begin{array}{l} \varphi \dots \text{IV. kv.} \\ \varphi = 2\pi - \varphi' = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} \end{array}$

$z = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$

[lze i $\cos\varphi = \frac{a}{|z|} = \frac{\sqrt{3}}{2} \quad \left[\begin{array}{l} \cos\varphi' = \frac{\sqrt{3}}{2} \\ \varphi' = 30° = \frac{\pi}{6} \end{array}\right]$ IV. kv. $\varphi = 2\pi - \frac{\pi}{6}$, $\varphi = \frac{11\pi}{6}$]

d) $z = -2 - 2i\sqrt{3} \quad [-2, -2\sqrt{3}]$

DÚ



$\varphi \dots$ III. kv.

$|z| = \sqrt{(-2)^2 + (-2\sqrt{3})^2}$

$|z| = \sqrt{4 + 4.3} = \sqrt{16} = 4$

$\sin\varphi = \frac{b}{|z|} = \frac{-2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2} \quad \left[\begin{array}{l} (*) \\ \sin\varphi' = \frac{\sqrt{3}}{2} \\ \varphi' = 60° = \frac{\pi}{3} \end{array}\right] \Rightarrow \begin{array}{l} \text{III. kv.} \\ \varphi = \pi + \varphi' \\ \varphi = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \end{array}$

$z = 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

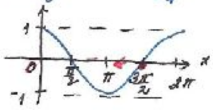
[lze i $\sin\varphi = \frac{b}{|z|} = -\frac{2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}$ $\ominus$ III, IV. kv.
$\cos\varphi = \frac{a}{|z|} = \frac{-2}{4} = -\frac{1}{2}$ $\ominus$ II, III. } III. kv. dále viz (*)
(neurčit si a oh. kvadrant)]

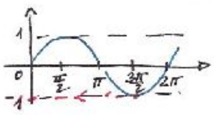
③ Vyjádřit v algebraickém tvaru

64/46 a) $z = 4\left(\cos\overset{30°}{\underbrace{\frac{\pi}{6}}_{\frac{\sqrt{3}}{2}}} + i\sin\overset{30°}{\underbrace{\frac{\pi}{6}}_{\frac{1}{2}}}\right) = 4\left(\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}\right) = 2\sqrt{3} + 2i$

b) $z = \frac{1}{2}\left(\cos\frac{7}{2}\pi + i\sin\frac{7}{2}\pi\right) = \frac{1}{2}\left(\underbrace{\cos\overset{270°}{\frac{3\pi}{2}}}_{0} + i\underbrace{\sin\overset{270°}{\frac{3\pi}{2}}}_{-1}\right) = \frac{1}{2}(0 + i(-1)) = -\frac{1}{2}i$

[$\frac{7}{2}\pi = \frac{7\pi}{2} - 2\pi = \frac{3\pi}{2} = 270°$
musíme získat úhel $0 - 2\pi$
$\left[\frac{7}{2}\pi = 3{,}5\pi = 3{,}5\pi - 2\pi = 1{,}5\pi = \frac{3}{2}\pi\right]$
$> 2\pi \Rightarrow$]

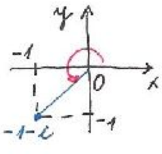




④ Vyjádřit v goniometrickém tvaru

DÚ a) $\frac{2}{-1+i} = \frac{2(-1-i)}{(-1+i)(-1-i)} = \frac{2(-1-i)}{1-\underbrace{i^2}_{-1}} = \frac{2(-1-i)}{2} = -1-i \quad [-1,-1]$

nejprve upr. na alg. tvar



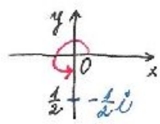
$|z| = \sqrt{(-1)^2 + (-1)^2}$
$|z| = 2$
$\varphi = 180° + 45°$
$\varphi = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$

$\Rightarrow z = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

b) $\frac{1}{2i} = \frac{1}{2i}\cdot\frac{(-i)}{(-i)} = \frac{-i}{\underbrace{-2i^2}_{+2}} = -\frac{i}{2} = 0 - \frac{1}{2}i \quad [0, -\frac{1}{2}]$

$\Rightarrow |z| = \sqrt{0^2 + (-\frac{1}{2})^2}$

$|z| = \sqrt{\frac{1}{4}} = \frac{1}{2}$

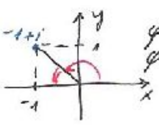


$\varphi = 270°$
$\varphi = \pi + \frac{\pi}{2}$
$\varphi = \frac{3\pi}{2}$

$\Rightarrow z = \frac{1}{2}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

c) $\frac{-3+i}{2+i} = \frac{(-3+i)}{2+i}\cdot\frac{(2-i)}{2-i} = \frac{-6+3i+2i-\overset{-1}{i^2}}{4-\underbrace{i^2}_{-1}} = \frac{-5+5i}{5} = \frac{5(-1+i)}{5} = -1+i \quad [-1, 1]$

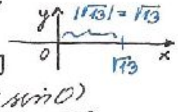
$\Rightarrow z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$



$\varphi = 180° - 45°$
$\varphi = \pi - \frac{\pi}{4}$
$\varphi = \frac{3\pi}{4}$

$|z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

d) $|3+2i| = \sqrt{9+4} = \sqrt{13}$
abs. hod. $\quad$ reál. č. $[\sqrt{13}, 0]$



$\varphi = 0$

$\Rightarrow z = \sqrt{13}(\cos 0 + i\sin 0)$

⑤ v algebraickém tvaru vyjádřete

DÚ 2.8 a) $2(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi) = 2(\underbrace{\cos 45°}_{\frac{\sqrt{2}}{2}} + i\underbrace{\sin 45°}_{\frac{\sqrt{2}}{2}}) = 2(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = \sqrt{2} + i\sqrt{2}$

b) $\frac{1}{2}(\cos 193\pi + i\sin 193\pi) = \frac{1}{2}(\underbrace{\cos\pi}_{-1} + i\underbrace{\sin\pi}_{0}) = \frac{1}{2}(-1 + 0i) = -\frac{1}{2}$

[$193\pi : 2\pi = \dots$ zb. 1π]   z grafu

c) $\cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi = \cos 60° + i\sin 60° = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

d) $\cos(-\frac{5}{3}\pi) + i\sin(-\frac{5}{3}\pi) = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \cos 60° + i\sin 60° = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

[$-\frac{5\pi}{3} \Rightarrow -1\frac{2}{3}\pi + 2\pi \Rightarrow \frac{\pi}{3}$] periodičnost

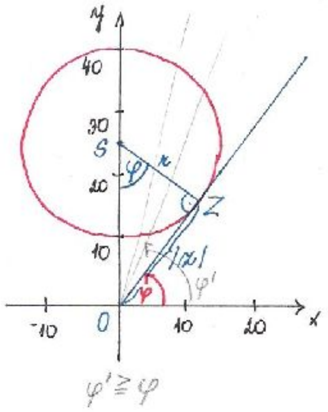
nebo $= \cos\frac{5\pi}{3} - i\sin\frac{5\pi}{3} = \cos 300° - i\sin 300° = \cos 60° - i(-\sin 60°) =$

využ. parity [$5\cdot\frac{\pi}{3} = 5\cdot 60° = 300°$] [⊕ ← IV. kv. → ⊖ ; $300° = 360° - \alpha$; $\alpha = 60°$] $= \cos 60° + i\sin 60° =$

$\cos(-x) = \cos x$ $\sin(-x) = -\sin x$ $= \frac{1}{2} + \frac{\sqrt{3}}{2}i$

⑥ Z komplexních čísel z, pro něž $|z - 25i| \leqq 15$ vyberte ta, která mají
Př/46 v $\langle 0, 2\pi)$ nejmenší argument.

nerovnici $|z - 25i| \leqq 15$ vyhovují všechna z ležící v kruhu $K(S, r = 15)$
$z_0 = 25i$ [0,25] $(\varphi' \geqq \varphi)$ $S[0,25]$



$\varphi' \geqq \varphi$

- nejmenší argument má to kompl. číslo, které leží na TEČNĚ KE KRUŽNICI vedené BODEM O[0,0]

$z = |z|(\cos\varphi + i\sin\varphi)$ $|z| = |OZ|$ $|OS| = 25$ $|SZ| = r = 15$

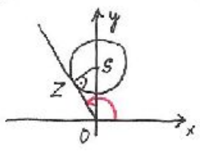
$\cos\varphi = \frac{|SZ|}{|OS|} = \frac{r}{25} = \frac{15}{25} = \frac{3}{5}$

$\sin\varphi = \frac{|OZ|}{|OS|} = \frac{20}{25} = \frac{4}{5}$

$|z| = |OZ| = \sqrt{|OS|^2 - |SZ|^2} = \sqrt{25^2 - 15^2} = \sqrt{625 - 225} = \sqrt{400} = 20$

$z = 20(\frac{3}{5} + \frac{4}{5}i) = 12 + 16i$

- pokud hl. kompl. čísla mající argument $\varphi = \frac{\pi}{2} \Rightarrow$ všechna $z = bi$, kde $10 \leqq b \leqq 40$
- pokud hl. kompl. č. vyhov. $|z - 25i| < 15 \Rightarrow$ s nejm. argum. neex. žádné
- pokud s největš. argumentem: totéž, ale 2. kvadrant $\Rightarrow \cos\varphi < 0$ ⊖, $\sin\varphi > 0$ ⊕



$z_{max} = 20(-\frac{3}{5} + \frac{4}{5}i) = -12 + 16i$

⑦ Jako př. ⑥

2.10 $|z + 25i| \leqq 15$

$z_0 = -25$ $r = 15$

$K[S, r = 15)$ $S[0, -25]$